On the Solution and Complexity of a Generalized Linear Complementarity Problem*

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(Received: 30 November 1992; accepted: 29 April 1993)

Abstract. We introduce some sufficient conditions under which a generalized linear complementarity problem (GLCP) can be solved as a pure linear complementarity problem. We also establish that the GLCP is in general a NP-Hard problem.

Key words. Complementarity problems, polynomial complexity, global optimization.

1. Introduction

In this paper we consider a generalized linear complementarity problem (GLCP) that consists of finding vectors u, v, z and y such that

$$u = p + Mz + Ny$$
$$v = q + Rz + Sy$$
$$u^T z = 0 , \quad u, v, z, y \geq 0 ,$$

where $u, z, p \in \mathbb{R}^n, y \in \mathbb{R}^m, v, q \in \mathbb{R}^l, M \in \mathbb{R}^{n \times n}, N \in \mathbb{R}^{n \times m}, R \in \mathbb{R}^{l \times n}$ and $S \in \mathbb{R}^{l \times m}$. This problem is a generalization of the well-known linear complementarity problem (LCP)

$$w = q + Mz , \quad w, z \geq 0 , \quad w^T z = 0 , \tag{1}$$

where M is a square matrix of order n and $w, z, q \in \mathbb{R}^n$. We recommend [2] for excellent discussions of the LCP.

The GLCP has found important applications in the solution of some global optimization problems by sequential techniques [4]. This problem has been mentioned in [9], where it is shown that it can be reduced to a concave nonlinear minimization problem. The existence of variables y and v without complementary makes this problem much harder to solve than the LCP. It is well known that if M is a positive semi-definite matrix, then the LCP can be solved in polynomial time [7]. However, we establish that the GLCP is NP-Hard even when M satisfies this property.

In this paper we discuss the solution of this problem by exploiting its reduction

* Support of this work has been provided by the Instituto Nacional de Investigação Científica de Portugal (INIC) under contract 89/EXA/5.

to the LCP. The first approach exploits the equivalence between the GLCP and a nonconvex quadratic program. As in [3], we establish a sufficient condition for a stationary point of the associated quadratic program to be a solution of the GLCP. The problem of finding such a point is in turn a LCP of the form (1). We also consider another reduction to a LCP of the form (1) under special properties of the matrices M, N, R and S and the vector q.

The organization of this paper is as follows. In Section 2 and 3 we establish the two existence results for this GLCP. The complexity result for the GLCP is proved in Section 4. Finally, some concluding remarks are presented in the last section of this paper.

2. Sufficient Matrices for the GLCP

In this section the following result given in [3] for the LCP is extended for the GLCP.

THEOREM 1. *If M is a row sufficient matrix, then the following condition holds*
- *if $(\bar{z}, \bar{\mu})$ is a* KKT *point of the quadratic program*

$$\begin{array}{ll} \min\limits_{z} & \frac{1}{2} z^T(M + M^T)z \\ \text{subject to} & Mz \geqslant -q \\ & z \geqslant 0 \end{array}$$

where $\bar{\mu}$ are the multipliers associated with the constraints $Mz \geqslant -q$, then $\bar{z}$ is a solution of the LCP.

The GLCP can be posed as a nonconvex quadratic minimization program

$$\begin{array}{ll} \min\limits_{u,v,z,y} & u^T z \\ \text{subject to} & u = p + Mz + Ny \\ & v = q + Rz + Sy \\ & u, v, z, y \geqslant 0 \,. \end{array}$$

We reformulate this program in its standard form

$$\min_{z,y} \quad \frac{1}{2}\begin{bmatrix} z \\ y \end{bmatrix}^T \begin{bmatrix} M + M^T & N \\ N^T & 0 \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} + \begin{bmatrix} p \\ 0 \end{bmatrix}^T \begin{bmatrix} z \\ y \end{bmatrix} \tag{2}$$

$$\text{subject to} \quad \begin{bmatrix} M & N \\ R & S \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} \geqslant \begin{bmatrix} -p \\ -q \end{bmatrix} \tag{3}$$

$$z, y \geqslant 0 \,. \tag{4}$$

A point $(\bar{z}, \bar{y}, \bar{\mu}_1, \bar{\mu}_2)$ is called a Karush–Kuhn–Tucker (KKT) point of the program (2)–(4) if it satisfies (3) and (4) and

$$\begin{bmatrix} p \\ 0 \end{bmatrix} + \begin{bmatrix} M+M^T & N \\ N^T & 0 \end{bmatrix}\begin{bmatrix} z \\ y \end{bmatrix} - \begin{bmatrix} M^T & R^T \\ N^T & S^T \end{bmatrix}\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \geqslant 0 \tag{5}$$

$$\begin{bmatrix} z \\ y \end{bmatrix}^T\left(\begin{bmatrix} p \\ 0 \end{bmatrix} + \begin{bmatrix} M+M^T & N \\ N^T & 0 \end{bmatrix}\begin{bmatrix} z \\ y \end{bmatrix} - \begin{bmatrix} M^T & R^T \\ N^T & S^T \end{bmatrix}\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}\right) = 0 \tag{6}$$

$$\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}^T\left(\begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} M & N \\ R & S \end{bmatrix}\begin{bmatrix} z \\ y \end{bmatrix}\right) = 0 \tag{7}$$

$$z, y, \mu_1, \mu_2 \geqslant 0$$

where $\mu_1 \in \mathbb{R}^n$ and $\mu_2 \in \mathbb{R}^l$.

In order to establish the main result of this section we require the following definition.

DEFINITION 1 [3]. A matrix $Z \in \mathbb{R}^{k\times k}$ is said to be column sufficient if it satisfies the implication

$$x_i(Zx)_i \leqslant 0 \text{ for all } i \in \{1, \dots, k\} \Rightarrow x_i(Zx)_i = 0 \text{ for all } i \in \{1, \dots, k\}\,.$$

The matrix Z is row sufficient if its transpose is column sufficient and is sufficient if it is both column and row sufficient.

The following proposition is stated in [2] and is used to establish our results.

PROPOSITION 1. *If $Z \in \mathbb{R}^{k\times k}$ is a row sufficient matrix, then*

(i) *all principal submatrices of Z are row sufficient*

(ii) *if $i, j \in \{1, \dots, k\}$* and $i \neq j$ *then*

$$(z_{ii} = 0 \textit{ and } z_{ji} \geqslant 0) \Rightarrow z_{ij} \leqslant 0\,.$$

As in [3], we investigate when a KKT point of the nonconvex quadratic program (2)–(4) is a solution of the GLCP. The following theorem provides a sufficient condition for this question.

THEOREM 2. *If*

$$W = \begin{bmatrix} M & 0 \\ R & 0 \end{bmatrix} \in \mathbb{R}^{(n+l)\times(n+l)}$$

is a row sufficient matrix, then the following condition holds

- *if $(\bar{z}, \bar{y}, \bar{\mu}_1, \bar{\mu}_2)$ is a* KKT *point of the quadratic program* (2)–(4) *then $(\bar{z}, \bar{y})$ is a solution of the* GLCP.

Proof. Since $(\bar{z}, \bar{y}, \bar{\mu}_1, \bar{\mu}_2)$ is a KKT point, then by (6) we have

$$\bar{z}^T(p + M\bar{z} + N\bar{y}) + \bar{z}^T(M^T\bar{z} - M^T\bar{\mu}_1 - R^T\bar{\mu}_2) = 0 \tag{8}$$

which by the feasibility constraints (3) and (4) implies

$$\bar{z}_i(M^T\bar{z} - M^T\bar{\mu}_1 - R^T\bar{\mu}_2)_i \leq 0\,, \quad i = 1, \ldots, n\,. \tag{9}$$

On the other hand, multiplying each of the first n rows of (5) by the correspondent components of $\bar{\mu}_1$, we get

$$(\bar{\mu}_1)_i(p + M\bar{z} + N\bar{y})_i + (\bar{\mu}_1)_i(M^T\bar{z} - M^T\bar{\mu}_1 - R^T\bar{\mu}_2)_i \geq 0\,, \quad i = 1, \ldots, n\,.$$

But from (7), we have

$$-(\bar{\mu}_1)_i(M^T\bar{z} - M^T\bar{\mu}_1 - R^T\bar{\mu}_2)_i \leq 0\,, \quad i = 1, \ldots, n\,. \tag{10}$$

Adding the two sets of inequalities (9) and (10) we obtain

$$\begin{aligned}&\bar{z}_i(M^T\bar{z})_i + (\bar{\mu}_1)_i(M^T\bar{\mu}_1)_i - \bar{z}_i(M\bar{\mu}_1)_i\\ &-(\bar{\mu}_1)_i(M^T\bar{z})_i - \bar{z}_i(R^T\bar{\mu}_2)_i + (\bar{\mu}_1)_i(R^T\bar{\mu}_2)_i \leq 0\,, \quad i = 1, \ldots, n\end{aligned}$$

or by using matrix notation

$$\left(\begin{bmatrix}\bar{z} - \bar{\mu}_1\\ -\bar{\mu}_2\end{bmatrix}\right)_i \left(\begin{bmatrix}M^T & R^T\\ 0 & 0\end{bmatrix}\begin{bmatrix}\bar{z} - \bar{\mu}_1\\ -\bar{\mu}_2\end{bmatrix}\right)_i \leq 0\,, \quad i = 1, \ldots, n + l\,. \tag{11}$$

By assumption W^T is a column sufficient matrix and the $n + l$ inequalities (11) and consequently the n inequalities (9) turn into equalities. Hence it follows by (8) that

$$\bar{z}^T(p + M\bar{z} + N\bar{y}) = 0$$

and this proves the theorem. □

Ye [10] has proved, using a different framework, that the generalized linear complementarity problem

$$Ax + By + Cz = q\,, \quad x, y, z \geq 0\,, \quad x^Ty = 0 \tag{12}$$

can be solved in polynomial time provided BA^T is a negative semi-definite matrix. It is a simple matter to see that the GLCP is a particular case of (12) where $BA^T = -W$. According to Ye's result, the GLCP can be solved in polynomial time provided W is a positive semi-definite matrix. Furthermore, in this last case W is row sufficient and we may solve the GLCP by computing a KKT point of the program (2)–(4).

Theorem 2 provides a sufficient condition that is not necessary. To show this, we introduce the following GLCP

$$\begin{bmatrix}u_1\\ u_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} + \begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}0\\ 1/3\end{bmatrix}y \tag{13}$$

$$v = [0] + [1 \quad 0]\begin{bmatrix}z_1\\ z_2\end{bmatrix} + [-1]y \tag{14}$$

$$u_1, u_2, v, z_1, z_2, y \geqslant 0$$
$$u_1 z_1 = u_2 z_2 = 0 . \tag{15}$$

It is a simple matter to see that the set defined by the linear constraints (13), (14) and (15) is

$$\{(u_1, 0, 0, 0, u_1, 0), u_1 \geqslant 0\}$$

and $u^T z = 0$ for all the points of this set. Thus all the feasible points of the associated quadratic program are stationary points and solutions of the GLCP. However, the matrix

$$W = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is not row sufficient, by Proposition 1 (ii).

As is discussed in [5], there exist some important global optimization problems leading to a GLCP in which the matrix M is positive semi-definite. If the GLCP does not contain any variables y, then it can be written in the following form

$$\begin{array}{ll} \min\limits_{z} & \frac{1}{2} z^T(M + M^T)z + p^T z \\ \text{subject to} & p + Mz \geqslant 0 \\ & q + Rz \geqslant 0 \\ & z \geqslant 0 . \end{array}$$

Hence the GLCP reduces to a convex quadratic program and can be solved in polynomial time. The same result holds if the GLCP contains some variables y and $R = 0$, since it is positive semi-definite the matrix W stated before. Next we establish the following result.

THEOREM 3. *If the matrix M is row sufficient then W is row sufficient if and only if $R = 0$.*

Proof. (i) If $R = 0$ and M is row sufficient, it is obvious that W is row sufficient.

(ii) Suppose now that W is row sufficient. Then $R \leqslant 0$ by Proposition 1 (ii). If $R \neq 0$, then there must exist at least one element r_{ij} of R such that $r_{ij} < 0$. From Proposition 1 (i) the matrix

$$\begin{bmatrix} m_{jj} & 0 \\ r_{ij} & 0 \end{bmatrix}$$

must be row sufficient. Hence

$$z_1(m_{jj} z_1 + r_{ij} z_2) < 0 \tag{16}$$

has to hold for all values of z_1 and z_2. However, by choosing $z_1 > 0$ and $z_2 \geqslant -(m_{jj} z_1^2)/(r_{ij} z_1)$ the condition (16) is violated and this completes the proof. □

By this theorem, if the GLCP contains some variables y and $R \neq 0$ there is no guarantee that a KKT point of the program (2)–(4) is a solution of the GLCP, even when M is a positive semi-definite matrix. So the GLCP seems to be much more difficult to solve in this case. The complexity of this GLCP will be discussed in Section 4.

3. Reduction into a Pure Linear Complementarity Problem

In this section we consider a GLCP in which N has at least a column, M is positive semi-definite and R contains a unique nonzero row. Hence the GLCP takes the form

$$\begin{aligned}
u &= p + Mz + Ny \\
v &= q + Sy \\
w_0 &= \lambda + r^T z + s^T y \\
u^T z &= 0\,, \quad u, v, w_0, z, y \geq 0
\end{aligned}$$

where $r \in \mathbb{R}^n, s \in \mathbb{R}^m, w_0, \lambda \in \mathbb{R}$ and M is a positive semi-definite matrix. We denote this generalized linear complementarity problem by GLCP1.

Let K be the set defined by

$$K = \{(z, y) \colon p + Mz + Ny \geq 0, q + Sy \geq 0, z, y \geq 0\}\,. \tag{17}$$

Let N_i be the i-th column of N and assume that the linear programs

$$h_i = \max_{(z,y)\in K} N_i^T z, \quad i = 1, \ldots, m$$

have optimal solutions. Let h be the vector whose components are h_i and consider the following linear complementarity problem (LCP1)

$$u = p + Mz + Ny - r\mu_0 \tag{18}$$

$$\eta = h - N^T z - s\mu_0 - S^T\theta \tag{19}$$

$$v = q + Sy \tag{20}$$

$$w_0 = \lambda + r^T z + s^T y \tag{21}$$

$$\begin{aligned}
&u, z, \eta, y, v, \theta, w_0, \mu_0 \geq 0 \\
&u^T z = \eta^T y = v^T\theta = w_0\mu_0 = 0\,.
\end{aligned} \tag{22}$$

Since M is a positive semi-definite matrix, then the matrix of this LCP1 also shares this property and this problem can be solved in polynomial time [7]. The next theorem provides a sufficient condition for which processing the LCP1 (find a solution or show that no solution exists) enables us to find a solution of the GLCP1 or to show that none exists.

THEOREM 4. (i) *If the* LCP1 *has no solution then the* GLCP1 *is infeasible.*
(ii) *Suppose that* $q \geq 0$ *and* $\max_{(y,z)\in K-\{0\}} s^T y < 0$. *If* $(\bar{u}, \bar{z}, \bar{\eta}, \bar{y}, \bar{v}, \bar{\theta}, \bar{w}_0, \bar{\mu}_0)$ *is a solution of* LCP1 *and* $\bar{y} \neq 0$, *then* $(\bar{u}, \bar{v}, \bar{z}, \bar{y}, \bar{w}_0)$ *is a solution of the* GLCP1.

Proof. (i) If the LCP1 has no solution then the set defined by its linear constraints (18)–(22) is empty [2]. Consequently this set is also empty for $\mu_0 = 0$ and $\theta = 0$. Since $N \leq 0$, then the set defined by the linear equations (18), (20), (21) and (22) is empty and the GLCP1 is infeasible.

(ii) If $\bar{\mu}_0 = 0$ the result follows immediately. Now suppose that $\bar{\mu}_0 > 0$. By the complementarity conditions we have $\bar{w}_0 = 0$, which implies

$$0 = \lambda + r^T \bar{z} + s^T \bar{y} .$$

On the other hand by (18), (19) and (20) the following conditions hold

$$\begin{aligned} 0 &= \bar{z}^T \bar{u} = p^T \bar{z} + \bar{z}^T M \bar{z} + \bar{z}^T N \bar{y} - \bar{\mu}_0 r^T \bar{z} \\ 0 &= \bar{y}^T \bar{\eta} = h^T \bar{y} - \bar{y}^T N \bar{z} - \bar{\mu}_0 s^T \bar{y} - \bar{y}^T S^T \bar{\theta} \\ 0 &= \bar{\theta}^T \bar{v} = q^T \bar{\theta} + \bar{\theta}^T S \bar{y} . \end{aligned} \tag{23}$$

Adding term by term these inequalities we get

$$p^T \bar{z} + h^T \bar{y} \bar{z}^T M \bar{z} - \bar{\mu}_0 (r^T \bar{z} + s^T \bar{y}) + q^T \bar{\theta} = 0$$

which by (23) is equivalent to

$$(-h^T + N^T \bar{z})^T \bar{y} = -\bar{\mu}_0 s^T \bar{y} + q^T \bar{\theta} .$$

Since $q \geq 0$ and $s^T \bar{y} < 0$, it is impossible to hold this inequality. Then $\bar{\mu}_0 = 0$ and $(\bar{u}, \bar{v}, \bar{z}, \bar{y}, \bar{w}_0)$ is a solution of the GLCP1. □

This theorem provides a sufficient condition for a GLCP in which N has at least a column, M is positive semi-definite and R contains a unique nonzero row to be solved in polynomial time. In particular a GLCP of this form can be solved in polynomial time if $q \geq 0$, $N \leq 0$, $s < 0$ and the solution of the LCP1 satisfies $\bar{y} \neq 0$.

Consider again the GLCP1 and assume that the hypotheses of Theorem 4(ii) hold. Furthermore let

$$K_1 = \{z \in \mathbb{R}^n : Mz \geq -p, r^T z \geq -\lambda\} .$$

If $K_1 \neq \emptyset$, a solution of the GLCP1 can be found by setting $y = 0$ and solving the following quadratic program

$$\min_{z \in K_1} p^T z + \frac{1}{2} z^T (M + M^T) z .$$

Now suppose that $K_1 = \emptyset$ and consider the linear program

$$\begin{array}{ll} \min\limits_{z,\theta,\mu_0} & \mu_0 \\ \text{subject to} & Mz - r\mu_0 \geqslant -p \\ & N^T z + s\mu_0 + S^T\theta \geqslant -h \\ & r^T z \geqslant -\lambda \\ & z, \theta, \mu_0 \geqslant 0 \,. \end{array} \tag{24}$$

Since $K_1 = \emptyset$, there are only two possible cases that are discussed below.

(i) If the linear program (24) is infeasible then either the LCP1 is infeasible, and the same occurs with the GLCP1, or the LCP1 must have a solution with $\bar{y} \neq 0$ and a solution of the GLCP1 is at hand, by Theorem 4(ii).

(ii) If the linear program (24) has an optimal solution $(\bar{z}, \bar{\mu}_0, \bar{\theta})$ with $\bar{\mu}_0 > 0$, then $z = \bar{z}$, $\mu_0 = \bar{\mu}_0$, $\theta = \bar{\theta}$ and $\bar{y} = 0$ satisfy the linear constraints of the LCP1. Since this LCP1 is monotone it must have at least a solution [2]. But y may be zero in all the solutions of the LCP1 and no conclusion may be stated about the existence of a solution to the GLCP1 in this case.

The preceding discussions indicates that the existence of a solution to the GLCP1 may be quite a hard problem even in the presence of the hypotheses of Theorem 4(ii). The situation is even worse when these hypotheses no longer hold. In the next section we show that the GLCP1 is in general a NP-Hard problem.

4. Complexity of the GLCP

It is well known that the LCP is in general a NP-Hard problem [1]. Despite this, there exist some classes of matrices for which the LCP can be solved in polynomial time ([2], [11]). In particular, a LCP with a positive semi-definite matrix M can be solved in polynomial time [7]. Since the LCP is a special case of the GLCP, then this latter problem is in general NP-Hard. In this section we show that the GLCP remains NP-Hard when its matrix M is positive semi-definite. As in [1] we use the subset sum problem.

- given $n+1$ positive integers $a_1, \dots, a_n$ and b, does $\Sigma_{i=1}^n a_i x_i = a^T x = b$ have an 0–1 solution?

Karp [6] has proved that this problem is NP-Complete. The subset sum problem can be seen as the problem of finding a feasible solution for the knapsack problem. This is in turn equivalent to check if the following concave quadratic program

$$\begin{array}{ll} \min\limits_{x} & x^T(e-x) \\ \text{subject to} & a^T x = b \\ & 0 \leqslant x \leqslant e \end{array} \tag{25}$$

has a global solution with zero value. Now we are able to prove the following theorem.

THEOREM 5. *If M is a positive semi-definite matrix then the GLCP is a NP-Hard problem.*

Proof. Consider the concave quadratic program (25). This problem equivalent to the bilinear program [8]

$$\begin{aligned} \min_{x,y} \quad & \frac{1}{2}e^T x + x^T(-I)y + \frac{1}{2}e^T y \\ \text{subject to} \quad & a^T x \leqslant b,\ -a^T x \leqslant -b,\ 0 \leqslant x \leqslant e \\ & a^T y \leqslant b,\ -a^T y \leqslant -b,\ 0 \leqslant y \leqslant e \end{aligned}$$

which in turn is equivalent to the following problem [5]

$$\min \quad \begin{bmatrix} b \\ -b \\ e \end{bmatrix}^T t + \frac{1}{2}e^T y$$

$$\text{subject to} \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e \\ b \\ -b \\ e \end{bmatrix} + \begin{bmatrix} 0 & a & -a & I \\ -a^T & 0 & 0 & 0 \\ a^T & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} -I \\ 0 \end{bmatrix} y \tag{26}$$

$$\gamma = \begin{bmatrix} b \\ -b \\ e \end{bmatrix} - \begin{bmatrix} a^T \\ -a^T \\ I \end{bmatrix} y \tag{27}$$

$$\alpha, \beta, \gamma, x, y, t, w_0 \geqslant 0 \tag{28}$$

$$\alpha^T x = \beta^T t = 0 .$$

Hence, finding a global solution with zero value of the quadratic program (25) reduces to solve the GLCP consisting of the constraints (26), (27) and (28) and

$$w_0 = -\begin{bmatrix} b \\ -b \\ e \end{bmatrix}^T t - \frac{1}{2}e^T y, \quad w_0 \geqslant 0 . \tag{29}$$

Hence the subset sum problem is equivalent to the GLCP (26), (27), (28) and (29). Therefore a GLCP with a positive semi-definite matrix M and $R \neq 0$ is a NP-Hard problem. □

5. Concluding Remarks

In this paper we have studied a generalized LCP (GLCP) that appears often in global optimization. We have shown that this problem is in general NP-Hard. We have also established two results that enable the solution of a generalized LCP as a pure linear complementarity problem of the form (1). We believe that these results may have some important applications in finding global solutions of some nonconvex optimization problems by the sequential LCP algorithm described in [4]. This will be a topic of our future research.

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